Dual group actions on C*–algebras and their description by Hilbert extensions

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February 1, 2008

Abstract

Given a C^*-algebra $\mathcal{A}$, a discrete abelian group $\mathcal{X}$ and a homomorphism $\Theta\colon\ \mathcal{X} \to \operatorname{Out}\mathcal{A}$, defining the dual action group $\Gamma \subset \operatorname{aut}\mathcal{A}$, the paper contains results on existence and characterization of Hilbert extensions of $\{\mathcal{A}, \Gamma\}$, where the action is given by $\hat{\mathcal{X}}$. They are stated at the (abstract) C*–level and can therefore be considered as a refinement of the extension results given for von Neumann algebras for example by Jones [16] or Sutherland [20, 21]. A Hilbert extension exists iff there is a generalized 2–cocycle. These results generalize those in [10], which are formulated in the context of superselection theory, where it is assumed that the algebra $\mathcal{A}$ has a trivial center, i.e. $\mathcal{Z} = \mathbb{C}\mathbb{1}$. In particular the well–known "outer characterization" of the second cohomology $H^2(\mathcal{X}, \mathcal{U}(\mathcal{Z}), \alpha_{\mathcal{X}})$ can be reformulated: there is a bijection to the set of all $\mathcal{A}$–module isomorphy classes of Hilbert extensions. Finally, a Hilbert space representation (due to Sutherland [20, 21] in the von Neumann case) is mentioned. The C*–norm of the Hilbert extension is expressed in terms of the norm of this representation and it is linked to the so–called regular representation appearing in superselection theory.

1 Introduction

In the Doplicher/Roberts theory (e.g. [12, 14]) it is a central assumption that the center of the C*–algebra $\mathcal{A}$ with which one starts the analysis is trivial, i.e. $\mathcal{Z} = \mathcal{Z}(\mathcal{A}) = \mathbb{C}\mathbb{1}$ (in a more categorial notation the assumption reads $(\iota, \iota) = \mathbb{C}\mathbb{1}$, where ι denotes the unit object of the strict monoidal C*–category, cf. [13]). From a systematical point of view it is interesting to study the properties and structural modifications of this theory if one assumes the presence of a nontrivial center $\mathcal{Z} \supset \mathbb{C}\mathbb{1}$. For example, if $(\mathcal{F}, \alpha_{\mathcal{G}})$ is a Hilbert C*–system for a compact group $\mathcal{G}$ and if the corresponding fixed point algebra $\mathcal{A}$ has a nontrivial center that satisfies the relation $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$, then the Galois correspondence does not hold anymore, i.e. we have the proper inclusion $\alpha_{\mathcal{G}} \subset \operatorname{stab}\mathcal{A}$ in $\operatorname{aut}\mathcal{F}$ (cf. [6, Section 7]). Recall, that in the trivial center situation it is a fundamental result of the theory that $\alpha_{\mathcal{G}} = \operatorname{stab}\mathcal{A}$. As a further justification we can also mention that in other generalizations of the Doplicher/Roberts theory as well as in some applications in mathematical physics a nontrivial center plays, to a certain extent, a distinguished role [17, 24, 15].

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In the present paper we continue the analysis of the presence of a nontrivial center in the construction of an extension algebra $\mathcal{F}$ (cf. [4, 5]). In particular, we study what we call dual group actions in the simple case where the group $\mathcal{X}$ is discrete and abelian (cf. with [10] in the special case where $\mathcal{Z} = \mathbb{C}\mathbb{1}$). This investigations will be done at the abstract C*–level which is the context of the Doplicher/Roberts theory mentioned above (cf. also [3]). On the other hand the results can be considered as a refinement of the study of twisted group algebras (twisted crossed products) on the concrete von Neumann algebra level (see e.g. [9, 16, 20, 21]). For example, the decisive C*–norm for the extension is defined intrisically and the natural representation (discussed e.g. by Sutherland) is related to the so–called regular representation that appears in the superselection theory [2]. We hope that the present analysis will be useful to obtain a more general 'inversion' theorem, where endomorphisms of $\mathcal{A}$ are involved. Indeed, the main theorems in Section 3 suggest that for a more general inversion theory in the nontrivial center situation the cohomological aspects may be essential.

The paper is structured in 5 sections: in the following section we will introduce the notion of a Hilbert C*–system and study some properties of the group homomorphism $\Theta\colon \mathcal{X} \to \mathrm{Out}\,\mathcal{A}$. Hilbert C*–systems are the result of the extension procedure mentioned above. In Section 3 we begin the study of the inverse (extension) problem: in particular it contains the result that a Hilbert extension exists iff there is a generalized 2–cocycle (to be defined there), and that in this case the set of all Hilbert extensions can be described in terms of the set of center–valued 2–cocycles of $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\alpha_\mathcal{X})$ (cf. Theorems 3.4 and 3.8). In the next section we relate the previously obtained results to the special case of the Doplicher/Roberts frame, where $\mathcal{Z} = \mathbb{C}\mathbb{1}$. Finally, in Section 5 we give a representation of the Hilbert extension, which was already studied by Sutherland [20, 21] in the von Neumann case. In particular, we show that if there is a faithful state of $\mathcal{A}$, this representation coincides with the so–called regular representation that appears in superselection theory (cf. e.g. [2]) and the intrinsic C*–norm turns out to be the operator norm of this representation.

2 Hilbert C*–systems

A C*-algebra $\mathcal{F}$ together with a pointwise norm-continuous group homomorphism $\mathcal{G} \ni g \to \alpha_g \in \mathrm{aut}\,\mathcal{F}$ of a locally compact group $\mathcal{G}$ is called a C*-system $\{\mathcal{F}, \alpha_\mathcal{G}\}$. Let $\mathcal{A} \subseteq \mathcal{F}$ be its fixed point algebra, i.e. $\mathcal{A} := \{A \in \mathcal{F} \mid \alpha_g A = A, g \in \mathcal{G}\}$. We denote by $\mathcal{A}^c := \mathcal{F} \cap \mathcal{A}' \subseteq \mathcal{F}$ the relative commutant of $\mathcal{A}$ w.r.t. $\mathcal{F}$. As is well-known, $\alpha_g \restriction \mathcal{A}^c$ is an automorphism of $\mathcal{A}^c$, so $\{\mathcal{A}^c, \alpha_\mathcal{G}\}$ is also a C*-system. We call it the *assigned* C*-system. The center $\mathcal{Z}(\mathcal{A})$ is denoted by $\mathcal{Z}$.

In the following let $\mathcal{G}$ be compact and abelian so that $\hat{\mathcal{G}} =: \mathcal{X}$ is abelian and discrete. The corresponding spectral projections w.r.t. $\{\mathcal{F}, \alpha_\mathcal{G}\}$ are denoted by Π_χ, $\chi \in \mathcal{X}$. Note that $\Pi_\iota \mathcal{F} = \mathcal{A}$, where ι is the unit element of $\mathcal{X}$.

2.1 Definition *A C*-system $\{\mathcal{F}, \alpha_\mathcal{G}\}$, $\mathcal{G}$ compact abelian, is called a* Hilbert C*-system *if* $\mathrm{spec}\,\alpha_\mathcal{G} = \mathcal{X}$ *and if each spectral subspace $\Pi_\chi \mathcal{F}$ contains a unitary U_χ, i.e. $\mathcal{U}(\Pi_\chi \mathcal{F}) \neq \emptyset$.*

If $\{\mathcal{F}, \alpha_\mathcal{G}\}$ is Hilbert, then $\beta_\chi := \mathrm{ad}\, U_\chi \restriction \mathcal{A}$ is an automorphism of $\mathcal{A}$, i.e. $\beta_\chi \in \mathrm{aut}\,\mathcal{A}$. We denote by π the canonical homomorphism of $\mathrm{aut}\,\mathcal{A}$ onto $\mathrm{Out}\,\mathcal{A} := \mathrm{aut}\,\mathcal{A}/\mathrm{int}\,\mathcal{A}$, where $\mathrm{int}\,\mathcal{A}$ denotes the normal subgroup of all inner automorphisms of $\mathcal{A}$. Then

$$\mathcal{X} \ni \chi \to \Theta(\chi) := \pi(\beta_\chi) \in \mathrm{Out}\,\mathcal{A} \tag{1}$$

is a group homomorphism of $\mathcal{X}$ into $\mathrm{Out}\,\mathcal{A}$, i.e. we have

2.2 Lemma *To each Hilbert C*–system $\{\mathcal{F}, \alpha_\mathcal{G}\}$, where $\mathcal{G}$ is compact abelian, there is canonically assigned a group homomorphism Θ: $\mathcal{X} \to \mathrm{Out}\,\mathcal{A}$ given by (1).*

Proof: Note that for $\chi_1, \chi_2 \in \mathcal{X}$ we have that $U_{\chi_1\chi_2} U^*_{\chi_2} U^*_{\chi_1} \in \mathcal{A}$ and this implies that $\beta_{\chi_1\chi_2} \circ \beta^{-1}_{\chi_2} \circ \beta^{-1}_{\chi_1} \in \mathrm{int}\,\mathcal{A}$. ■

We mention next the characterization of those Hilbert C*-systems where Θ is an isomorphism and of those where the classes $\Theta(\chi)$ are pairwise disjoint. Recall that $\alpha, \beta \in \mathrm{aut}\,\mathcal{A}$ are called disjoint if

$$(\alpha, \beta) := \{X \in \mathcal{A} \mid X\alpha(A) = \beta(A)X \quad \text{for all} \quad A \in \mathcal{A}\} = 0\,.$$

2.3 Proposition *(i) Θ is a monomorphism iff no spectral subspace $\Pi_\chi \mathcal{A}^c, \chi \neq \iota$, of the assigned C*–system contains a unitary.*

(ii) The classes $\Theta(\chi)$ are pairwise disjoint iff $\mathcal{A}^c = \mathcal{Z}$, i.e. the relative commutant coincides with the center of $\mathcal{A}$.

Proof: For one of the directions of part (i) take a unitary $U_\chi \in \Pi_\chi(\mathcal{A}^c)$ with $\iota \neq \chi \in \mathcal{X}$, so that the corresponding $\beta_\chi = \mathrm{id}$ and $\pi(\beta_\chi) = \mathrm{int}\,\mathcal{A}$. Thus Θ is not injective. For the other implication take $\mathcal{X} \ni \chi_0 \neq \iota$ with $\chi_0 \in \ker\Theta$, i.e. $\Theta(\chi_0) = \mathrm{int}\,\mathcal{A}$. Thus there exists a unitary $V \in \mathcal{U}(\mathcal{A})$ with $\mathrm{ad}\,V = \mathrm{ad}\,U_{\chi_0}$. From this we get $V^* U_{\chi_0} \in \mathcal{U}(\mathcal{A}^c) \cap \Pi_{\chi_0}(\mathcal{F})$, i.e. $\Pi_{\chi_0}\mathcal{A}^c \neq \emptyset$.

Finally, part (ii) follows from [7, Lemma 10.1.8]. ■

We mention several useful concepts for Hilbert C*-systems $\{\mathcal{F}, \alpha_\mathcal{G}\}$ with a compact abelian group.

2.4 Definition *$\beta \in \mathrm{aut}\,\mathcal{A}$ is called a* canonical automorphism *if $\beta := \mathrm{ad}\,V \restriction \mathcal{A}\,, V \in \bigcup_{\chi\in\mathcal{X}} \mathcal{U}(\Pi_\chi \mathcal{F})$. The set of all canonical automorphisms is denoted by Γ.*

2.5 Remark Note that for the set of canonical automorphisms we have $\mathrm{int}\,\mathcal{A} \subseteq \Gamma \subseteq \mathrm{aut}\,\mathcal{A}$ and that for $\alpha, \beta \in \Gamma$ the automorphisms $\alpha \circ \beta$ and $\beta \circ \alpha$ are unitarily equivalent. Furthermore, $\mathcal{X} \cong \Gamma/\mathrm{int}\mathcal{A}$ and the set Γ is sometimes called *dual action* on $\mathcal{A}$.

For any $\gamma_1, \gamma_2 \in \Gamma$ we write

$$\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1} \circ \gamma_2^{-1} = \mathrm{ad}\,\epsilon(\gamma_1, \gamma_2),$$

where $\epsilon(\gamma_1, \gamma_2) \in \mathcal{U}(\mathcal{A})$ and the class $\widehat{\epsilon}(\gamma_1, \gamma_2) := \epsilon(\gamma_1, \gamma_2) \bmod \mathcal{U}(\mathcal{Z})$ is uniquely defined.

2.6 Lemma *The permutators $\epsilon(\cdot, \cdot)$ satisfy the following relations:*

$$\begin{aligned}
\epsilon(\gamma_1, \gamma_2)\epsilon(\gamma_2, \gamma_1) &\equiv \mathbb{1} \bmod \mathcal{U}(\mathcal{Z})\,, && \gamma_1, \gamma_2 \in \Gamma\,, \\
\epsilon(\iota, \gamma) \equiv \epsilon(\gamma, \iota) &\equiv \mathbb{1} \bmod \mathcal{U}(\mathcal{Z})\,, && \gamma \in \Gamma\,, \\
\gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3) &\equiv \epsilon(\gamma_1\gamma_2, \gamma_3) \bmod \mathcal{U}(\mathcal{Z})\,, && \gamma_1, \gamma_2, \gamma_3 \in \Gamma\,, \\
A\gamma_1(B)\epsilon(\gamma_1, \gamma_2) &\equiv \epsilon(\gamma_1', \gamma_2')B\gamma_2(A) \bmod \mathcal{U}(\mathcal{Z})\,, && \gamma_1, \gamma_2, \gamma_1', \gamma_2' \in \Gamma \text{ and} \\
& && A \in (\gamma_1, \gamma_1') \cap \mathcal{U}(\mathcal{A})\,,\ B \in (\gamma_2, \gamma_2') \cap \mathcal{U}(\mathcal{A}).
\end{aligned}$$

Proof: The first and second equations above are obvious. To prove the third one consider the the inner automorphism characterized by the l.h.s. of the equation:

$$\begin{aligned}
\mathrm{ad}\Big(\gamma_1(\epsilon(\gamma_2, \gamma_3))\epsilon(\gamma_1, \gamma_3)\Big) &= \mathrm{ad}\Big(\gamma_1(\epsilon(\gamma_2, \gamma_3))\Big) \circ \mathrm{ad}\Big(\epsilon(\gamma_1, \gamma_3)\Big) \\
&= \gamma_1\,\mathrm{ad}(\epsilon(\gamma_2, \gamma_3))\,\gamma_1^{-1} \circ \mathrm{ad}(\epsilon(\gamma_1, \gamma_3)) \\
&= \gamma_1\,(\gamma_2\gamma_3\gamma_2^{-1}\gamma_3^{-1})\,\gamma_1^{-1}\,(\gamma_1\gamma_3\gamma_1^{-1}\gamma_3^{-1}) = (\gamma_1\gamma_2)\,\gamma_3\,(\gamma_1\gamma_2)^{-1}\,\gamma_3^{-1} \\
&= \mathrm{ad}\Big(\epsilon(\gamma_1\gamma_2, \gamma_3)\Big)\,,
\end{aligned}$$

and this shows the desired relation. Finally, to prove the last equation recall that from the assumptions we have $\gamma_1' = \mathrm{ad}(A) \circ \gamma_1$ and $\gamma_2' = \mathrm{ad}(B) \circ \gamma_2$. From this we compute

$$\begin{aligned} \mathrm{ad}(\epsilon(\gamma_1',\gamma_2')) &= (\mathrm{ad}(A)\circ\gamma_1)\circ(\mathrm{ad}(B)\circ\gamma_2)\circ(\mathrm{ad}(A)\circ\gamma_1)^{-1}\circ(\mathrm{ad}(B)\circ\gamma_2)^{-1} \\ &= \mathrm{ad}(A)\circ\mathrm{ad}(\gamma_1(B))\circ\underbrace{\gamma_1\circ\gamma_2\circ\gamma_1^{-1}\circ\gamma_2^{-1}}_{\mathrm{ad}(\epsilon(\gamma_1,\gamma_2))}\circ\mathrm{ad}(\gamma_2(A))^{-1}\circ\mathrm{ad}(B)^{-1}\,. \end{aligned}$$

Therefore we get

$$\mathrm{ad}\Big(\epsilon(\gamma_1',\gamma_2')B\gamma_2(A)\Big) = \mathrm{ad}\Big(A\gamma_1(B)\epsilon(\gamma_1,\gamma_2)\Big)$$

which implies the last equation of the statement. ■

2.7 Definition *Let $\beta_\chi \in \Theta(\chi)$, $\chi \in \mathcal{X}$, with $\beta_\iota = id_{\mathcal{A}}$, be a system of representatives, i.e. $\pi(\beta_\chi) = \Theta(\chi)$. Then $\beta_{\mathcal{X}}$ is called a* lifting *of Θ if $\mathcal{X} \ni \chi \to \beta_\chi \in aut\,\mathcal{A}$ is a homomorphism.*

2.8 Remark For the notion of lifting see for example Jones [16]. Sutherland [20, 21] says that Θ *splits* if there is a lifting of Θ. If Θ is an isomorphism then a lifting is also called *monomorphic section* (this latter name is used by Doplicher/Haag/Roberts [10]).

Results on the existence of liftings when $\mathcal{A}$ is a von Neumann algebra and in a more general context w.r.t. the group $\mathcal{X}$ (theory of Q-kernels) are due to Sutherland [20, 21]. Further, recall also the result of Doplicher/Haag/Roberts [10] in the "automorphism case" of the superselection theory, where $\mathcal{Z} = \mathbb{C}\,\mathbb{1}$ and $\mathcal{A}$ is a so-called quasilocal algebra w.r.t. a net of local von Neumann algebras (see also [2]).

3 Hilbert extensions

The question concerning the description of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ by $\mathcal{A}$ and 'something else' is called the *reconstruction problem*. It is posed, for example, by Takesaki [23, p. 202] and by Bratteli/Robinson [8, p. 137]. Also the superselection structures in algebraic quantum field theory are connected with the reconstruction problem (for the automorphism case see Doplicher/Haag/Roberts [10]).

From Lemma 2.2 it seems natural to consider the corresponding *inverse problem*, which is an extension problem. This is just the emphasis in the mentioned papers by Sutherland and Jones (see also Nakamura/Takeda [19, 22]) as well as an essential aspect of the superselection theory (cf. [10, 2]).

3.1 Definition *Let a system $\{\mathcal{A}, \Theta(\mathcal{X})\}$ be given where $\mathcal{X}$ is a discrete abelian group and where Θ: $\mathcal{X} \to Out\,\mathcal{A}$ is a homomorphism and put $\mathcal{G} := \hat{\mathcal{X}}$. A Hilbert C^*-system $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ is called a* Hilbert extension *of $\{\mathcal{A}, \Theta(\mathcal{X})\}$ if $\mathcal{A} = \Pi_\iota\mathcal{F}$ and $\Theta(\mathcal{X})$ coincides with the homomorphism given by Lemma 2.2.*

Now let $\{\mathcal{A}, \Theta(\mathcal{X})\}$ and $\mathcal{G}$ be given as in the previous definition. As it is pointed out, for example in [16], a crucial object for the extension problem is the so-called *obstruction* $\mathrm{Ob}\,\Theta$. We recall the relevant relations: Choose a system $\beta_\chi \in \Theta(\chi)$, $\chi \in \mathcal{X}, \beta_\iota := \mathrm{id}_{\mathcal{A}}$ of representatives. Then

$$\beta_{\chi_1} \circ \beta_{\chi_2} = \mathrm{ad}\,(\omega(\chi_1,\chi_2)) \circ \beta_{\chi_1\chi_2}, \tag{2}$$

where

$$\mathcal{X} \times \mathcal{X} \ni (\chi_1,\chi_2) \to \omega(\chi_1,\chi_2) \in \mathcal{U}(\mathcal{A}) \tag{3}$$

and we have the intertwining property

$$\omega(\chi_1, \chi_2) \in (\beta_{\chi_1\chi_2}, \beta_{\chi_1} \circ \beta_{\chi_2}), \tag{4}$$

which is implied by (2). Moreover we have

$$\omega(\iota, \chi) = \omega(\chi, \iota) = \mathbb{1}. \tag{5}$$

Now associativity yields

$$\mathrm{ad}\,(\omega(\chi_1, \chi_2)\omega(\chi_1\chi_2, \chi_3)) = \mathrm{ad}\,(\beta_{\chi_1}(\omega(\chi_2, \chi_3))\omega(\chi_1, \chi_2\chi_3))$$

so that there is $\gamma(\chi_1, \chi_2, \chi_3) \in \mathcal{U}(\mathcal{Z})$ with

$$\omega(\chi_1, \chi_2)\omega(\chi_1\chi_2, \chi_3) = \gamma(\chi_1, \chi_2, \chi_3)\beta_{\chi_1}(\omega(\chi_2, \chi_3))\omega(\chi_1, \chi_2\chi_3).$$

If $\gamma(\chi_1, \chi_2, \chi_3) = \mathbb{1}$ for all $\chi_1, \chi_2, \chi_3 \in \mathcal{X}$ we obtain the equation

$$\omega(\chi_1, \chi_2)\omega(\chi_1\chi_2, \chi_3) = \beta_{\chi_1}(\omega(\chi_2, \chi_3))\omega(\chi_1, \chi_2\chi_3). \tag{6}$$

Obviously, the existence of a system of representatives $\beta_{\mathcal{X}}$ such that equation (6) has a solution ω equipped with the properties (3)–(5) is necessary for the existence of a Hilbert extension. Even more, the existence of such a solution is also sufficient for the existence of a Hilbert extension.

3.2 Definition *A function ω, assigned to a given system $\beta_{\mathcal{X}}$ of representatives of $\Theta(\mathcal{X})$, equipped with the properties (3)–(6) is called a generalized 2-cocycle.*

One calculates easily that the existence of a generalized 2-cocycle is independent of the choice of the system $\beta_{\mathcal{X}}$ of representatives. Further, a generalized cocycle ω for $\beta_{\mathcal{X}}$ satisfies the relation

$$\mathrm{ad}\,(\omega(\chi_1, \chi_2)\omega(\chi_2, \chi_1)^{-1}) = \beta_{\chi_1} \circ \beta_{\chi_2} \circ \beta_{\chi_1}^{-1} \circ \beta_{\chi_2}^{-1}.$$

The existence of a lifting of Θ can be expressed in terms of generalized 2-cocycles as follows.

3.3 Lemma *There exists a lifting $\beta_{\mathcal{X}}$ of Θ iff to each system $\gamma_{\mathcal{X}}$ of representatives there corresponds a generalized 2–cocycle ω of the form*

$$\omega(\chi_1, \chi_2) \equiv \gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1\chi_2} \quad \mathrm{mod}\,\mathcal{U}(\mathcal{Z})\,,$$

where $V_\chi \in \mathcal{U}(\mathcal{A})$, $V_\iota = \mathbb{1}$. In this case, i.e. if there is a lifting $\beta_{\mathcal{X}}$, then a corresponding generalized 2–cocycle ω is given by $\omega(\chi_1, \chi_2) = \mathbb{1}$ for all $\chi_1, \chi_2 \in \mathcal{X}$.

Proof: Let $\beta_\chi = \mathrm{ad}(V_\chi) \circ \gamma_\chi$, $V_\chi \in \mathcal{U}(\mathcal{A})$, $\chi \in \mathcal{X}$. Now if $\omega(\chi_1, \chi_2) = \gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1\chi_2}Z$ for some $Z \in \mathcal{U}(\mathcal{Z})$, then we have on the one hand $\beta_{\chi_1\chi_2} = \mathrm{ad}(V_{\chi_1\chi_2}) \circ \gamma_{\chi_1\chi_2}$ and on the other

$$\beta_{\chi_1} \circ \beta_{\chi_2} = (\mathrm{ad}(V_{\chi_1}) \circ \gamma_{\chi_1}) \circ (\mathrm{ad}(V_{\chi_2}) \circ \gamma_{\chi_2}) = \mathrm{ad}\Big(V_{\chi_1}\gamma_{\chi_1}(V_{\chi_2})\,\omega(\chi_1, \chi_2)\Big) \circ \gamma_{\chi_1\chi_2}\,,$$

which using the assumption on ω and the fact that $\mathrm{ad}(V_{\chi_1\chi_2}Z) = \mathrm{ad}(V_{\chi_1\chi_2})$, implies that $\beta_{\chi_1\chi_2} = \beta_{\chi_1} \circ \beta_{\chi_2}$, i.e. there is a lift of Θ. To prove the converse let $\beta_{\chi_1\chi_2} = \beta_{\chi_1} \circ \beta_{\chi_2}$, so that from the above relations we have

$$\mathrm{ad}(V_{\chi_1\chi_2}) = \mathrm{ad}\Big(V_{\chi_1}\gamma_{\chi_1}(V_{\chi_2})\,\omega(\chi_1, \chi_2)\Big)\,,$$

which implies $\omega(\chi_1, \chi_2) = \gamma_{\chi_1}(V_{\chi_2}^{-1})V_{\chi_1}^{-1}V_{\chi_1\chi_2}\,\mathrm{mod}\,\mathcal{U}(\mathcal{Z})$. ∎

3.4 Theorem *Let ω be a generalized 2–cocycle for the system $\beta_{\mathcal{X}}$ of representatives. Then there is a Hilbert extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Theta(\mathcal{X})\}$.*

Proof: The proof consists of several steps that correspond to gradually imposing a richer structure on an initially considered $\mathcal{A}$–left module:

1. Indeed, choose first system of 1-dimensional linear spaces, generated by abstract elements U_χ, $\chi \in \mathcal{X}$, $U_\iota := \mathbb{1} \in \mathcal{A}$. Form the $\mathcal{A}$–left modules $\mathcal{A} \otimes \mathbb{C}U_\chi$ and $\mathcal{F}_0 := \bigoplus_\chi (\mathcal{A} \otimes \mathbb{C}U_\chi)$. By identification $A \otimes \mathbb{1} \leftrightarrow A$, $\mathbb{1} \otimes U_\chi \leftrightarrow U_\chi$ one has

$$\mathcal{F}_0 = \left\{ \sum_{\chi\,,\ \text{finite sum}} A_\chi U_\chi \mid \quad A_\chi \in \mathcal{A} \right\},$$

where $\{U_\chi \mid \chi \in \mathcal{X}\}$ forms an abstract $\mathcal{A}$–module basis.

2. Next we want to equip $\mathcal{F}_0$ with a multiplication structure. First $\mathcal{F}_0$ becomes an $\mathcal{A}$–bimodule extending linearly the following definition

$$U_\chi\, A := \beta_\chi(A)\, U_\chi\,, \quad A \in \mathcal{A}\,, \chi \in \mathcal{X}\,,$$

where $\beta_{\mathcal{X}}$ is the system of representatives to which we associate the generalized cocycle ω. Now the product structure is finally specified by putting

$$U_{\chi_1} \cdot U_{\chi_2} := \omega(\chi_1, \chi_2)\, U_{\chi_1\chi_2}\,, \quad \chi_1, \chi_2 \in \mathcal{X}\,,$$

where the cocycle equation (6) guarantees that the product is associative and the boundary conditions (5) lead to $U_\chi \cdot \mathbb{1} = \mathbb{1} \cdot U_\chi = U_\chi$. Note that the preceding product structure already implies that the U_χ are invertible. Indeed, it can be checked easily that the inverse is given explicitly by

$$U_\chi^{-1} := \beta_{\chi^{-1}}\Big(\omega(\chi, \chi^{-1})^{-1}\Big) U_{\chi^{-1}}$$

(use for example the relation $\beta_\chi(\omega(\chi^{-1}, \chi)) = \omega(\chi, \chi^{-1})$, which follows from the cocycle equation (6) by putting $\chi_1 := \chi$, $\chi_2 := \chi^{-1}$ and $\chi_3 = \chi$).

3. The following step consists in defining a *–structure on $\mathcal{F}_0$. This is done by putting

$$U_\chi^* := \omega(\chi^{-1}, \chi)^* U_{\chi^{-1}} \quad \text{and} \quad (AU_\chi)^* := U_\chi^* A^*.$$

We still have to check that this definition is consistent, in particular with the product structure in $\mathcal{F}_0$, i.e. we have to verify:

$$(U_\chi^*)^* = U_\chi\,, \quad (U_\chi A)^* = A^* U_\chi^* \quad \text{and} \quad (U_{\chi_1} \cdot U_{\chi_2})^* = U_{\chi_2}^* \cdot U_{\chi_1}^*\,. \tag{7}$$

For the first equation we have

$$\begin{aligned}
(U_\chi^*)^* &= \Big(\omega(\chi^{-1}, \chi)^*\, U_{\chi^{-1}}\Big)^* = U_{\chi^{-1}}^*\, \omega(\chi^{-1}, \chi) = \omega(\chi, \chi^{-1})^*\, U_\chi\, \omega(\chi^{-1}, \chi) \\
&= \omega(\chi, \chi^{-1})^*\, \beta_\chi\Big(\omega(\chi^{-1}, \chi)^*\Big) U_\chi = \omega(\chi, \chi^{-1})^*\, \omega(\chi, \chi^{-1})\, U_\chi \\
&= U_\chi
\end{aligned}$$

The second equation in (7) can also be checked immediately from the definitions considered above. For the last equation we will consider the two sides separately: for the r.h.s. we have

$$\begin{aligned}
U_{\chi_2}^* \cdot U_{\chi_1}^* &= \omega(\chi_2^{-1}, \chi_2)^*\, U_{\chi_2^{-1}} \cdot \omega(\chi_1^{-1}, \chi_1)^*\, U_{\chi_1^{-1}} \\
&= \omega(\chi_2^{-1}, \chi_2)^*\, \beta_{\chi_2^{-1}}\Big(\omega(\chi_1^{-1}, \chi_1)^*\Big) U_{\chi_2^{-1}}\, U_{\chi_1^{-1}} \\
&= \omega(\chi_2^{-1}, \chi_2)^*\, \beta_{\chi_2^{-1}}\Big(\omega(\chi_1^{-1}, \chi_1)^*\Big) \omega(\chi_2^{-1}, \chi_1^{-1})\, U_{(\chi_1\chi_2)^{-1}} \\
&= \omega(\chi_2^{-1}, \chi_2)^*\, \omega((\chi_1\chi_2)^{-1}, \chi_1)^*\, \underbrace{\omega(\chi_2^{-1}, \chi_1^{-1})^* \omega(\chi_2^{-1}, \chi_1^{-1})}_{\mathbb{1}}\, U_{(\chi_1\chi_2)^{-1}}\,,
\end{aligned}$$

where we have used the relation

$$\beta_{\chi_2^{-1}}(\omega(\chi_1^{-1},\chi_1)) = \omega(\chi_2^{-1},\chi_1^{-1})\,\omega(\chi_2^{-1}\chi_1^{-1},\chi_1)\,,$$

which again follows from the cocycle equation (6) taking now $\chi_1 := \chi_2^{-1}$, $\chi_2 := \chi_1^{-1}$ and $\chi_3 = \chi_1$. Now the l.h.s. reads

$$\begin{aligned}(U_{\chi_1}\cdot U_{\chi_2})^* &= U^*_{\chi_1\chi_2}\,\omega(\chi_1,\chi_2)^* = \omega((\chi_1\chi_2)^{-1},\chi_1\chi_2)^*\,U_{(\chi_1\chi_2)^{-1}}\,\omega(\chi_1,\chi_2)^* \\ &= \omega((\chi_1\chi_2)^{-1},\chi_1\chi_2)^*\beta_{(\chi_1\chi_2)^{-1}}\Big(\omega(\chi_1,\chi_2)^*\Big)\,U_{(\chi_1\chi_2)^{-1}}\,.\end{aligned}$$

Thus to show the last equation in (7) we need to prove that

$$\omega((\chi_1\chi_2)^{-1},\chi_1\chi_2)^*\,\beta_{(\chi_1\chi_2)^{-1}}\Big(\omega(\chi_1,\chi_2)^*\Big) = \omega(\chi_2^{-1},\chi_2)^*\,\omega\Big((\chi_1\chi_2)^{-1},\chi_1\Big)^*$$

or taking adjoints

$$\beta_{(\chi_1\chi_2)^{-1}}\Big(\omega(\chi_1,\chi_2)\Big)\,\omega((\chi_1\chi_2)^{-1},\chi_1\chi_2) = \omega\Big((\chi_1\chi_2)^{-1},\chi_1\Big)\,\omega(\chi_2^{-1},\chi_2)\,.$$

But the preceding equation is nothing else than the cocycle equation (7) with $\chi_1 := (\chi_1\chi_2)^{-1}$, $\chi_2 := \chi_1$ and $\chi_3 := \chi_2$. Finally, note that since $\beta_{\chi^{-1}}\Big(\omega(\chi,\chi^{-1})^{-1}\Big) = \omega(\chi^{-1},\chi)^*$ we also have that the U_χ, are unitary, i.e. $U^*_\chi = U^{-1}_\chi$, $\chi \in \mathcal{X}$.

4. Here we will define a representation of the compact abelian group $\mathcal{G} = \widehat{\mathcal{X}}$ in terms of automorphisms of the *–algebra $\mathcal{F}_0$. The automorphisms are fixed by putting

$$\alpha_g(U_\chi) := \chi(g)\,U_\chi \quad\text{and}\quad \alpha_g(AU_\chi) := A\,\alpha_g(U_\chi) = \chi(g)\,A\,U_\chi\,,\quad g\in\mathcal{G},\ A\in\mathcal{A},\ \chi\in\mathcal{X}\,.$$

First we check that with the definition above the α_g is indeed an automorphism compatible with the structure in $\mathcal{F}_0$:

$$\begin{aligned}\alpha_g\Big(U_{\chi_1}U_{\chi_2}\Big) &= \alpha_g\Big(\omega(\chi_1,\chi_2)\,U_{\chi_1\chi_2}\Big) = (\chi_1\chi_2)(g)\,\omega(\chi_1,\chi_2)\,U_{\chi_1\chi_2} \\ &= \chi_1(g)\chi_2(g)\,U_{\chi_1}\,U_{\chi_2} = \alpha_g\Big(U_{\chi_1}\Big)\alpha_g\Big(U_{\chi_2}\Big)\end{aligned}$$

and

$$\begin{aligned}\alpha_g\Big(U^*_\chi\Big) &= \alpha_g\Big(\omega(\chi^{-1},\chi)^*\,U_{\chi^{-1}}\Big) = (\chi^{-1})(g)\,\omega(\chi^{-1},\chi)^*\,U_{\chi^{-1}} \\ &= \overline{\chi}(g)\,U^*_\chi = \alpha_g\Big(U_\chi\Big)^*\,.\end{aligned}$$

It can be also easily seen that the assignment $\mathcal{G}\ni g\to\alpha_g\in\operatorname{aut}\mathcal{F}_0$ is an injective group homomorphism. Finally, note that the fixed point algebra of the previous action coincides with $\mathcal{A}$, i.e. for $F\in\mathcal{F}_0$, $\alpha_g(F) = F$ for all $g\in\mathcal{G}$ iff $F\in\mathcal{A}$. Indeed, for an arbitrary element $\sum_\chi A_\chi U_\chi\in\mathcal{F}_0$ the equation $\sum_\chi \chi(g)A_\chi U_\chi = \sum_\chi A_\chi U_\chi$, $g\in\mathcal{G}$, implies by the base property of the U_χ that $\chi(g)A_\chi = A_\chi$, $g\in\mathcal{G}$, $\chi\in\mathcal{X}$. Therefore if $\chi_0\neq\iota$, then there is a $g_0\in\mathcal{G}$ with $\chi_0(g_0)\neq 1$ and this shows that $A_{\chi_0} = 0$. The converse implication is obvious.

5. Finally, to specify a C*–norm on $\mathcal{F}_0$ we introduce the following $\mathcal{A}$–valued scalar product (note the variation w.r.t. the definition in [2, p. 101]):

$$\langle F_1,F_2\rangle := \sum_\chi \beta^{-1}_\chi(A^*_\chi B_\chi)\,,\quad\text{where}\quad F_1 = \sum_\chi A_\chi U_\chi\,,\ F_2 = \sum_\chi B_\chi U_\chi\in\mathcal{F}_0.$$

This scalar product satisfies the properties

$$\langle F_1,F_2\rangle^* = \langle F_2,F_1\rangle\,,\quad \langle F_1,F_1\rangle\geq 0\quad\text{and}\quad \langle F_1,F_1\rangle = 0 \text{ iff } F_1 = 0\,.$$

Next we show that

$$\langle F_1, F_2 \rangle = \Pi_\iota(F_1^* F_2)\,,$$

Indeed, using the definitions above we have

$$F_1^* F_2 = \sum_{\chi_1,\chi_2} U_{\chi_1}^* A_{\chi_1}^* B_{\chi_2} U_{\chi_2} = \sum_{\chi_1,\chi_2} \omega(\chi_1^{-1},\chi_1)^*\, \beta_{\chi_1^{-1}}(A_{\chi_1}^* B_{\chi_2})\, \omega(\chi_1^{-1},\chi_2) U_{\chi_1^{-1}\chi_2}\,.$$

Putting, $\chi_1 = \chi_2 = \chi$ in the preceding expression we get

$$\begin{aligned}
\Pi_\iota(F_1^* F_2) &= \sum_\chi \omega(\chi^{-1},\chi)^*\, \beta_{\chi^{-1}}(A_\chi^* B_\chi)\, \omega(\chi^{-1},\chi) \\
&= \sum_\chi \omega(\chi^{-1},\chi)^* \omega(\chi^{-1},\chi)\, \beta_\chi^{-1}(A_\chi^* B_\chi)\, \omega(\chi^{-1},\chi)^* \omega(\chi^{-1},\chi) \\
&= \langle F_1, F_2 \rangle\,,
\end{aligned}$$

where for the second equation before we have used eq. (2) in the form $\beta_{\chi^{-1}} = \mathrm{ad}\,(\omega(\chi^{-1},\chi)) \circ \beta_\chi^{-1}$. In particular the relation above implies the following invariance property: $\langle \alpha_g(F_1), \alpha_g(F_2) \rangle = \langle F_1, F_2 \rangle$, $g \in \mathcal{G}$.

Define next the following norm on $\mathcal{F}_0$ by

$$|F| := \| \langle F, F \rangle \|^{\frac{1}{2}}\,, \quad F \in \mathcal{F}_0\,,$$

and the representation of $\mathcal{F}_0$ on $(\mathcal{F}_0, |\cdot|)$ in terms of multiplication operators

$$\rho(F) X := F X\,, \quad F, X \in \mathcal{F}_0\,.$$

Note that by the definition of the $\mathcal{A}$–valued scalar product the property $\rho(F^*) = \rho(F)^*$, $F \in \mathcal{F}_0$, holds. Now using the corresponding operator norm we introduce

$$\|F\|_* := |\rho(F)|_{op}\,, \quad F \in \mathcal{F}_0\,,$$

which by similar arguments as in [2, p. 102-103] satisfies the C*–property $\|F^* F\|_* = \|F\|_*^2$. Further, it satisfies also (cf. again the previous reference)

$$\|A\|_* = \|A\|\,, \quad A \in \mathcal{A} \quad \text{and} \quad \|\alpha_g(F)\|_* = \|F\|_*\,, \quad g \in \mathcal{G}\,,\, F \in \mathcal{F}_0\,.$$

Therefore, we can finally extend α_g isometrically from $\mathcal{F}_0$ to

$$\mathcal{F} := \mathrm{clo}_{\|\cdot\|_*}(\mathcal{F}_0)\,.$$

Further, $\alpha_{\mathcal{G}} \subset \mathrm{aut}\, \mathcal{F}$ is norm continuous w.r.t. the pointwise norm convergence, because for any $F_0 = \sum_\chi A_\chi U_\chi \in \mathcal{F}_0$ we have

$$\|\alpha_{g_1}(F_0) - \alpha_{g_2}(F_0)\|_* = \|\sum_\chi \Big(\chi(g_1) - \chi(g_2)\Big) A_\chi U_\chi\|_* \leq \sum_\chi |\chi(g_1) - \chi(g_2)|\, \|A_\chi\|\,.$$

By construction we also have that $U_\chi \in \Pi_\chi(\mathcal{F})$, $\chi \in \mathcal{X}$. Therefore from the definitions of Sections 2 and 3 we have constructed a Hilbert C*–extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Gamma\}$ and the proof is concluded. ■

Using now Lemma 3.3 one has

3.5 Corollary *If there is a lifting of Θ, then there is a Hilbert extension of $\{\mathcal{A}, \Theta(\mathcal{X})\}$, corresponding to $\omega = \mathbb{1}$.*

3.6 Remark The construction in the proof of the previous theorem generalizes to the nontrivial center situation the procedure already presented (with small modifications) in [2, Section 3.6].

The second problem consists in the description of all Hilbert extensions. For this purpose let $\Omega(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ be the set of all $\mathcal{U}(\mathcal{Z})$–valued 2–cocycles λ, i.e. λ satisfies equation (6) and condition (5), but (3),(4) are replaced by $\lambda(\chi_1,\chi_2)\in\mathcal{U}(\mathcal{Z})$. For example, $\lambda(\chi_1,\chi_2):=\mathbb{1}$ for all $\chi_1,\chi_2\in\mathcal{X}$ is such a cocycle. Further let $\Omega_0(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ be the set of all $\mathcal{U}(\mathcal{Z})$–valued coboundaries ∂Z, i.e.

$$\partial Z(\chi_1,\chi_2):=\frac{Z(\chi_1)\beta_{\chi_1}(Z(\chi_2))}{Z(\chi_1\chi_2)}\,,$$

where $Z(\cdot)$ is a $\mathcal{U}(\mathcal{Z})$-valued 1-cycle, $Z(\iota)=\mathbb{1}$. Then ∂Z is a $\mathcal{U}(\mathcal{Z})$-valued 2-cocycle, $\Omega\supseteq\Omega_0$. As usual, Ω and Ω_0 are abelian groups w.r.t. pointwise multiplication and the second cohomology is given by $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}}):=\Omega/\Omega_0$.

Next we need the concept of *$\mathcal{A}$–module isomorphism* of Hilbert extensions.

3.7 Definition *Let $\{\mathcal{F}^1,\alpha^1_{\mathcal{G}}\}$, $\{\mathcal{F}^2,\alpha^2_{\mathcal{G}}\}$ be Hilbert extensions of $\{\mathcal{A},\Theta(\mathcal{X})\}$. They are called $\mathcal{A}$–module isomorphic if there is an algebraic isomorphism Φ: $\mathcal{F}^1\to\mathcal{F}^2$, with $\Phi(A)=A$ for all $A\in\mathcal{A}$ and $\Phi\circ\alpha^1_g=\alpha^2_g\circ\Phi$ for all $g\in\mathcal{G}$.*

3.8 Theorem *Let ω_0 be a generalized 2–cocycle. Then:*

(i) Each $\mathcal{U}(\mathcal{Z})$–valued 2–cocycle λ yields a Hilbert extension generated by the generalized 2–cocycle $\omega:=\lambda\cdot\omega_0$ and each Hilbert extension is generated by some $\mathcal{U}(\mathcal{Z})$–valued 2–cocycle λ via $\omega:=\lambda\cdot\omega_0$.

(ii) Two Hilbert extensions are $\mathcal{A}$–module isomorphic iff the generating generalized 2–cocycles ω_1,ω_2 differ only by a $\mathcal{U}(\mathcal{Z})$–valued coboundary ∂Z, i.e. $\omega_1=\partial Z\cdot\omega_2$.

Proof: (i) If two generalized 2–cocycles ω_1,ω_2 are given, then note first that $\lambda(\chi_1,\chi_2):=\omega_1(\chi_1,\chi_2)\omega_2(\chi_1,\chi_2)^{-1}\in\mathcal{U}(\mathcal{Z})$ for all χ_1,χ_2, because of condition (4). Further, eq. (5) follows from the corresponding properties of ω_1 and ω_2. Finally, the cocycle equation for $\lambda(\chi_1,\chi_2)$ is a consequence of the following computation:

$$\begin{aligned}
\lambda(\chi_1,\chi_2)\lambda(\chi_1\chi_2,\chi_3) &= \omega_1(\chi_1,\chi_2)\omega_2(\chi_1,\chi_2)^{-1}\cdot\omega_1(\chi_1\chi_2,\chi_3)\omega_2(\chi_1\chi_2,\chi_3)^{-1}\\
&= \omega_1(\chi_1,\chi_2)\omega_1(\chi_1\chi_2,\chi_3)\omega_2(\chi_1\chi_2,\chi_3)^{-1}\omega_2(\chi_1,\chi_2)^{-1}\\
&= (\omega_1(\chi_1,\chi_2)\omega_1(\chi_1\chi_2,\chi_3))\cdot(\omega_2(\chi_1,\chi_2)\omega_2(\chi_1\chi_2,\chi_3))^{-1}\\
&= \beta_{\chi_1}(\omega_1(\chi_2,\chi_3))\omega_1(\chi_1,\chi_2\chi_3)\cdot(\beta_{\chi_1}(\omega_2(\chi_2,\chi_3))\omega_2(\chi_1,\chi_2\chi_3))^{-1}\\
&= \beta_{\chi_1}(\omega_1(\chi_2,\chi_3))\omega_1(\chi_1,\chi_2\chi_3)\omega_2(\chi_1,\chi_2\chi_3)^{-1}\beta_{\chi_1}(\omega_2(\chi_2,\chi_3))^{-1}\\
&= \beta_{\chi_1}(\omega_1(\chi_2,\chi_3)\omega_2(\chi_2,\chi_3)^{-1})\omega_1(\chi_1,\chi_2\chi_3)\omega_2(\chi_1,\chi_2\chi_3)^{-1}\\
&= \beta_{\chi_1}(\lambda(\chi_2,\chi_3))\cdot\lambda(\chi_1,\chi_2\chi_3),
\end{aligned}$$

i.e. if one fixes a generalized 2-cocycle ω_0, then $\omega:=\lambda\cdot\omega_0$ runs through all generalized 2–cocycles ω if λ runs through all $\mathcal{U}(\mathcal{Z})$–valued 2–cocycles in $\Omega(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$.

(ii) Let $\{\mathcal{F}^1,\alpha^1_{\mathcal{G}}\}$ and $\{\mathcal{F}^2,\alpha^2_{\mathcal{G}}\}$ be two Hilbert extensions of $\{\mathcal{A},\Theta(\mathcal{X})\}$ and denote the corresponding set of abstract unitaries by $\{U_\chi\mid\chi\in\mathcal{X}\}$ resp. $\{V_\chi\mid\chi\in\mathcal{X}\}$.

Suppose first that there exists coboundary $\partial Z\in\Omega_0(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$, where $\beta_{\mathcal{X}}$ is system of representatives in Θ, such that the corresponding generalized cocycles ω_1 and ω_2 satisfy $\omega_1=\partial Z\cdot\omega_2$. In this case we will show that the extensions are isomorphic. Indeed, define the isomorphism by

$$\Phi(AU_\chi):=A\,Z(\chi)\,V_\chi\,,\quad A\in\mathcal{A}\,,\ \chi\in\mathcal{X}\,,$$

and extend it by linearity to the corresponding left $\mathcal{A}$–module. Now Φ is even a *–homomorphism between the *–algebras $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ that are defined in step 3 of the proof of Theorem 3.4. This follows from the following computations:

$$\begin{aligned}
\Phi(U_\chi A) &= \Phi\Big(\beta_\chi(A)U_\chi\Big) = Z(\chi)\, V_\chi A = \Phi(U_\chi)\Phi(A)\,, \\
\Phi(U_\chi U_{\chi'}) &= \Phi\Big(\omega_1(\chi,\chi')U_{\chi\chi'}\Big) = \partial Z(\chi,\chi')\cdot\omega_2(\chi,\chi')\, Z(\chi\chi')\, V_{\chi\chi'} \\
&= \frac{Z(\chi)\beta_\chi(Z(\chi'))}{Z(\chi\chi')}\cdot Z(\chi\chi')\, V_\chi V_{\chi'} = Z(\chi)V_\chi\, Z(\chi')V_{\chi'} = \Phi(U_\chi)\Phi(U_{\chi'})\,, \\
\Phi(U_\chi^*) &= \Phi\Big(\omega_1(\chi^{-1},\chi)^* U_{\chi^{-1}}\Big) = \partial Z(\chi^{-1},\chi)^*\cdot\omega_2(\chi^{-1},\chi)^* Z(\chi^{-1})V_{\chi^{-1}} \\
&= Z(\chi^{-1})^*\beta_{\chi^{-1}}(Z(\chi))^* Z(\chi^{-1})\,\omega_2(\chi^{-1},\chi)^* V_{\chi^{-1}} = (Z(\chi)V_\chi)^* = \Phi(U_\chi)^*\,,
\end{aligned}$$

where $\chi,\chi'\in\mathcal{X}$, $A\in\mathcal{A}$. Note further that on $\mathcal{F}_0^1$ we already have $\Phi\circ\alpha_g^1 = \alpha_g^2\circ\Phi$, $g\in\mathcal{G}$, since for any $\chi\in\mathcal{X}$ we have

$$\Phi\circ\alpha_g^1(AU_\chi) = \chi(g)\, A\, Z(\chi)V_\chi = \alpha_g^2(A\, Z(\chi)V_\chi) = \alpha_g^2\circ\Phi(AU_\chi)\,.$$

Recall that Φ is a bijection between $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ and we will finish this part of the proof if we can also show that Φ is even an isometry w.r.t the corresponding C*–norms, because in this case we can isometrically extend Φ to the desired Hilbert extension isomorphism $\Phi\colon\ \mathcal{F}^1\to\mathcal{F}^2$. Now denote by $\langle\cdot,\cdot\rangle_k$ the $\mathcal{A}$–valued scalar products on $\mathcal{F}_0^k$, $k=1,2$, given in step 5 of the proof of Theorem 3.4. For any $F=\sum_\chi A_\chi U_\chi\in\mathcal{F}_0^1$, so that $\Phi(F)=\sum_\chi A_\chi\, Z(\chi)\, V_\chi\in\mathcal{F}_0^2$, we have the following invariance

$$\langle\Phi(F),\Phi(F)\rangle_2 = \sum_\chi \beta_\chi^{-1}\Big(Z(\chi)^* A_\chi^* A_\chi Z(\chi)\Big) = \sum_\chi \beta_\chi^{-1}\Big(A_\chi^* A_\chi\Big) = \langle F,F\rangle_1\,.$$

From this and recalling the definition of the C*–norm again in step 5 of the proof of Theorem 3.4 we immediately get the desired isometry property:

$$\|\Phi(F)\|_* = \sup_{\substack{X_2\,\in\,\mathcal{F}_0^2\\ |X_2|\,\le\,1}} |\Phi(F)X_2| = \sup_{\substack{X_1\,\in\,\mathcal{F}_0^1\\ |X_1|\,\le\,1}} |\Phi(F)X_1| = \|F\|_*\,.$$

To prove the converse implication assume that $\Phi\colon\ \mathcal{F}_1\to\mathcal{F}_2$ specifies the isomorphy of the Hilbert extensions. Use the unitaries $\{U_\chi\mid\chi\in\mathcal{X}\}$ and $\{V_\chi\mid\chi\in\mathcal{X}\}$ in $\mathcal{F}_1$ resp. $\mathcal{F}_2$ to define the unitary

$$Z(\chi) := \Phi(U_\chi)\, V_\chi^*\,,\quad \chi\in\mathcal{X}\,,$$

that satisfies $Z(\iota)=\mathbb{1}$. Even more $Z(\chi)\in\mathcal{U}(\mathcal{Z})$, since for any $A\in\mathcal{A}$ we have

$$A\, Z(\chi) = \Phi(AU_\chi)\, V_\chi^* = \Phi\Big(U_\chi\beta_\chi^{-1}(A)\Big)V_\chi^* = \Phi(U_\chi)\,(A^*V_\chi)^* = Z(\chi)\, A\,.$$

Finally, for $\chi,\chi'\in\mathcal{X}$ we have

$$\begin{aligned}
Z(\chi\chi') &= \Phi\Big(\omega_1(\chi,\chi')^{-1}U_\chi U_{\chi'}\Big)\cdot V_{\chi'}^* V_\chi^*\,(\omega_2(\chi,\chi')^{-1})^* \\
&= \omega_1(\chi,\chi')^{-1}\,\Phi(U_\chi)\, Z(\chi')\, V_\chi^*\,\omega_2(\chi,\chi') \\
&= \omega_1(\chi,\chi')^{-1}\,\Phi(U_\chi)\Big(\beta_\chi(Z(\chi')^*)V_\chi\Big)^*\omega_2(\chi,\chi') \\
&= \omega_1(\chi,\chi')^{-1}\; Z(\chi)\,\beta_\chi(Z(\chi'))\,\omega_2(\chi,\chi')\,.
\end{aligned}$$

Now recalling the definition of the coboundary ∂Z, the preceding equations imply that $\omega_1(\chi,\chi') = \partial Z(\chi,\chi')\cdot\omega_2(\chi,\chi')$, $\chi,\chi'\in\mathcal{X}$, and the prove is concluded. ■

3.9 Remark (i) Note that the results are independent of the choice of the system $\beta_{\mathcal{X}}$ of representatives of $\Theta(\mathcal{X})$. Theorem 3.8 means that there is a bijection between $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ and the set of all $\mathcal{A}$–module isomorphy classes of Hilbert extensions of $\{\mathcal{A},\Theta(\mathcal{X})\}$ if there is one extension. In other words, the theorem gives an *outer* characterization of $H^2(\mathcal{X},\mathcal{U}(\mathcal{Z}),\beta_{\mathcal{X}})$ by the set of all $\mathcal{A}$–module isomorphy classes of Hilbert extensions.

(ii) For a closer analysis of the second cohomology in the special cases were $\Gamma \cong \mathbb{Z}_N$ and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ see [1]. Consider also the abstract results in [18, Chapter 4].

4 The case of a trivial center

In this case we have $\mathcal{Z} = \mathbb{C}\mathbb{1}$, thus $\mathcal{U}(\mathcal{Z}) = \mathbb{T}\mathbb{1}$ and this implies that two automorphisms $\alpha, \beta \in \Gamma$ are either unitarily equivalent or otherwise disjoint. The following result is a special case of the famous Doplicher/Roberts theorem (see [13, 3]) in the present automorphism context.

4.1 Proposition *If there is a system of representatives $\epsilon(\alpha,\beta)$ of the permutator classes $\widehat{\epsilon}(\alpha,\beta)$ which satisfy the equations*

$$
\begin{aligned}
\epsilon(\gamma_1,\gamma_2)\epsilon(\gamma_2,\gamma_1) &= \mathbb{1}\,, \\
\epsilon(\iota,\gamma) = \epsilon(\gamma,\iota) &= \mathbb{1}\,, \\
\gamma_1(\epsilon(\gamma_2,\gamma_3))\epsilon(\gamma_1,\gamma_3) &= \epsilon(\gamma_1\gamma_2,\gamma_3)\,, \\
A\beta_{\chi_1}(B)\epsilon(\chi_1,\chi_2) &= \epsilon'(\chi_1,\chi_2)B\beta_{\chi_2}(A)\,,
\end{aligned}
$$

for all $A \in (\beta_{\chi_1},\beta'_{\chi_1})$, $B \in (\beta_{\chi_2},\beta'_{\chi_2})$, where ϵ' belongs to $\beta'_{\mathcal{X}}$, then there is a generalized 2–cocycle ω_0 w.r.t. some system β_χ of representatives of the classes $\chi \in \Gamma/\mathrm{int}\mathcal{A}$, with

$$\omega_0(\chi_1,\chi_2)\omega_0(\chi_2,\chi_1)^{-1} = \epsilon(\beta_{\chi_1},\beta_{\chi_2}).$$

In this case there is a Hilbert extension $\mathcal{F}$ of $\{\mathcal{A},\Gamma\}$.

Conversely, if there is a Hilbert extension $\mathcal{F}$ of $\{\mathcal{A},\Gamma\}$, then to each $\alpha \in \Gamma$ there corresponds a unitary $V_\alpha \in \bigcup_{\chi\in\mathcal{X}} \mathcal{U}(\Pi_\chi\mathcal{F})$, such that $\alpha = ad\,V_\alpha\restriction\mathcal{A}$ and

$$\epsilon(\alpha,\beta) := V_\alpha\,V_\beta\,V_\alpha^{-1}\,V_\beta^{-1}\,,$$

is a system of representatives of the permutators $\widehat{\epsilon}(\alpha,\beta)$ satisfying the equations above.

4.2 Remark (i) In the present case the 2-cocycles λ of the preceding section are $\mathbb{T}\mathbb{1}$-valued and the relation (6) becomes the usual cocycle equation

$$\lambda(\chi_1,\chi_2)\lambda(\chi_1\chi_2,\chi_3) = \lambda(\chi_2,\chi_3)\lambda(\chi_1,\chi_2\chi_3).$$

(ii) In the particular case where $\mathcal{A}$ is the inductive limit of a net of von Neumann algebras (which is a standard situation in algebraic quantum field theory, $\mathcal{A}$ being the so–called quasilocal algebra) it can be shown that there is a lift $\gamma_{\mathcal{X}}$ of a given system of representatives $\beta_{\mathcal{X}}$, $\beta_\chi \in \chi$ (cf. Definition 2.7), and by Corollary 3.5 we have that $\omega(\chi_1,\chi_2) = 1$ is an admissible 2–cocycle of the system $\gamma_{\mathcal{X}}$. For a detailed construction of the lift see [10], [2, Section 3.2].

5 A Hilbert space representation of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$

Following Sutherland [20, 21] one can introduce a faithful Hilbert space representation of a Hilbert extension $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ of $\{\mathcal{A}, \Theta(\mathcal{X})\}$.

First let $\mathcal{H}$ be a Hilbert space and let π be a faithful representation of $\mathcal{A}$ on $\mathcal{H}$. Form the Hilbert space $\mathcal{K} := l^2(\mathcal{X}, \mathcal{H})$ by completion of $C_0(\mathcal{X} \to \mathcal{H})$ w.r.t. the norm $\|f\|^2 := \sum_\chi \|f(\chi)\|^2_{\mathcal{H}}$. Choose a system $\beta(\mathcal{X})$ of representatives of $\Theta(\mathcal{X})$ and let ω be a corresponding generalized 2-cocycle such that $U_{\chi_1} \cdot U_{\chi_2} = \omega(\chi_1, \omega_2) U_{\chi_1 \chi_2}$. Now define a representation Φ of $\mathcal{F}_0 \subset \mathcal{F}$ on $\mathcal{K}$ by

$$\begin{aligned}
(\Phi(A)f)(\chi) &:= \pi(\beta_{\chi^{-1}}(A)) f(\chi), \quad A \in \mathcal{A}, \\
\Phi(U_{\chi_0})f)(\chi) &:= \pi(\omega(\chi^{-1}, \chi_0)) f(\chi_0^{-1}\chi), \quad \chi_0 \in \mathcal{X}, \\
\Phi(AU_\chi) &:= \Phi(A)\Phi(U_\chi), \quad A \in \mathcal{A},\, \chi \in \mathcal{X}.
\end{aligned}$$

Note that $\Phi(\mathbb{1}) = \mathbb{1}_{\mathcal{K}}$ and $\|\Phi(A)\|_{\mathcal{K}} = \|A\|$. One calculates easily

$$\begin{aligned}
\Phi(U_{\chi_1})\Phi(U_{\chi_2}) &= \Phi(\omega(\chi_1, \chi_2))\Phi(U_{\chi_1\chi_2}), \\
\Phi(U_\chi)\Phi(A) &= \Phi(\beta_\chi(A))\Phi(U_\chi), \\
\Phi(A^*) = \Phi(A)^*, \quad \Phi(U_\chi^*) &= \Phi(U_\chi)^*.
\end{aligned}$$

Further $\Phi(\sum_\chi A_\chi U_\chi) = 0$ implies $\sum_\chi A_\chi U_\chi = 0$, i.e. Φ is a *-isomorphism from $\mathcal{F}_0$ onto $\Phi(\mathcal{F}_0) \subset \mathcal{L}(\mathcal{K})$. Recall that

$$\|\Phi(F)\|_{\mathcal{K}} = \sup_{\|f\| \le 1} \|\Phi(F)f\|_{\mathcal{K}}.$$

We have

5.1 Lemma *The relation*

$$\sup_{g \in \mathcal{G}} \|\Phi(\alpha_g F)\|_{\mathcal{K}} < \infty, \quad F \in \mathcal{F}_0, \tag{8}$$

holds.

Proof: With $F = \sum_\chi A_\chi U_\chi$ we have

$$\begin{aligned}
\|\Phi(F)f\|^2 &= \sum_{y \in \mathcal{X}} \| \sum_\chi \pi(\alpha_{y^{-1}}(A_\chi)\omega(y^{-1}, \chi)) f(y^{-1}\chi)\|^2_{\mathcal{H}} \\
&\le \sum_{y \in \mathcal{X}} (\sum_\chi \|\pi(\alpha_{y^{-1}}(A_\chi)\omega(y^{-1}, \chi)) f(y^{-1}\chi)\|)^2 \\
&\le \sum_{y \in \mathcal{X}} (\sum_\chi \|A_\chi\| \cdot \|f(y^{-1}\chi)\|)^2 \le \sum_{y \in \mathcal{X}} (\sum_\chi \|A_\chi\|^2)(\sum_\chi \|f(y^{-1}\chi)\|^2) \\
&= (\sum_\chi \|A_\chi\|^2) \sum_\chi \sum_{y \in \mathcal{X}} \|f(y^{-1}\chi)\|^2 = N(F)\|f\|^2 \sum_\chi \|A_\chi\|^2,
\end{aligned}$$

where $N(F)$ denotes the number of terms of F. Hence we obtain

$$\|\Phi(F)\|_{\mathcal{K}} \le N(F)^{1/2} (\sum_\chi \|A_\chi\|^2)^{1/2} =: C_F.$$

and this implies

$$\|\Phi(\alpha_g F)\|_{\mathcal{K}} \le C_F, \quad g \in \mathcal{G},$$

because the number of terms of $\alpha_g F$ equals that of F and $\|\chi(g)A_\chi\| = \|A_\chi\|$. This implies the inequality (8). ■

This result means that

$$\|\Phi(F)\|_{sup} := \sup_{g \in \mathcal{G}} \|\Phi(\alpha_g F)\|_{\mathcal{K}}$$

is a C*-norm on $\mathcal{F}_0$.

5.2 Theorem *The relation*

$$\|\Phi(F)\|_{sup} = \|F\|_*, \quad F \in \mathcal{F}_0\,,$$

holds, and in particular $\|\Phi(F)\|_{\mathcal{K}} \leq \|F\|_*$, $F \in \mathcal{F}_0$.

Proof: The norm $\mathcal{F}_0 \ni F \to \|\Phi(F)\|_{sup}$ has the properties $\|\Phi(A)\|_{sup} = \|A\|$ for all $A \in \mathcal{A}$ and $\|\Phi(\alpha_g F)\|_{sup} = \|\Phi(F)\|_{sup}$ for all $g \in \mathcal{G}$. However, according to Doplicher/Roberts [11, p. 105] there is at most one C*-norm on $\mathcal{F}_0$ with the mentioned properties. ■

5.3 Remark If there is a faithful state ϕ_0 of $\mathcal{A}$, then Theorem 5.2 can be improved. In this case

$$\|\Phi(F)\|_{\mathcal{K}} = \|F\|_*, \quad F \in \mathcal{F}_0,$$

holds. This is implied by the fact that in this case Sutherland's representation Φ of $\mathcal{F}_0$ on $\mathcal{K}$ is unitarily equivalent to the so–called regular representation of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ (restricted to $\mathcal{F}_0$) given by the (faithful) GNS-representation π of $\{\mathcal{F}, \alpha_{\mathcal{G}}\}$ on the GNS-Hilbert space $\mathcal{H}_\pi$ w.r.t. the $\mathcal{G}$-invariant state $\phi(F) := \phi_0(\Pi_\iota F)$, $F \in \mathcal{F}$, such that $\|\Phi(F)\|_{\mathcal{K}} = \|\pi(F)\|_{\mathcal{H}_\pi}$ for all $F \in \mathcal{F}_0$, but $\|\pi(F)\|_{\mathcal{H}_\pi} = \|F\|_*$ for all $F \in \mathcal{F}$ (see, for example, [2, p. 108 ff.]).

Acknowledgements One of us (H.B.) wants to thank Alan Carey for his kind invitation to the University of Adelaide as well as for valuable suggestions of an earlier version of the manuscript. The other author (F.Ll.) expresses his gratitude to Sergio Doplicher for his hospitality at the 'Dipartamento di Matematica dell' Università di Roma 'La Sapienza" in october '99. The visit was supported by a EU TMR network "Implementation of concept and methods from Non–Commutative Geometry to Operator Algebras and its applications", contract no. ERB FMRX-CT 96-0073.

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